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SHORT COMMUNICATION

Analytical solutions of couple stress fluid flows with slip boundary conditions

M. Devakar [a,*], D. Sreenivasu [b], B. Shankar [c]

[a] Department of Mathematics, Visvesvaraya National Institute of Technology Nagpur, Nagpur 440 010, India
[b] Department of Mathematics, CVR College of Engineering, Hyderabad 501 510, India
[c] Department of Mathematics, Osmania University, Hyderabad 500 007, India

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Abstract In the present article, the exact solutions for fundamental flows namely Couette, Poiseuille and generalized Couette flows of an incompressible couple stress fluid between parallel plates are obtained using slip boundary conditions. The effect of various parameters on velocity for each problem is discussed. It is found that, for each of the problems, the solution in the limiting case as couple stresses approaches to zero is similar to that of classical viscous Newtonian fluid. The results indicate that, the presence of couple stresses decreases the velocity of the fluid.

1. Introduction

The flows of non-Newtonian fluids have many practical applications in modern technology and industries, led various researchers to attempt diverse flow problems related to several non-Newtonian fluids. One such fluid that has attracted the attention of numerous researchers in fluid mechanics during the last five decades is the theory of couple stress fluids proposed by Stokes [1]. Couple stress fluid theory is a simple generalization of the classical theory of viscous Newtonian fluids that allow the sustenance of couple stresses and body couples in the fluid medium. The concept of couple stresses arises due to the way in which the mechanical interactions in the fluid medium are modeled. The stress tensor here is not symmetric. This theory adequately describes the flow behavior of fluids containing a substructure such as lubricants with polymer additives, liquid crystals and animal blood [2,3]. Many studies have been made on the hydrodynamic lubrication of squeeze film flows considering the lubricant as a couple stress fluid and the studies revealed that the couple stress fluid increases the load carrying capacity of the journal bearing [4–7].

A majority of flows of Newtonian and non-Newtonian fluids have been studied under no-slip boundary condition. However, this condition might not always hold, and that the fluid slippage might occur at the solid boundaries [8–11]. The works of O'Neill et al. [12] and Basset [13] indicate the existence of slip at the solid boundary. In fact, in early 19th century Navier [14] proposed a general boundary condition that presents the possibility of slip at the solid boundary. This boundary condition assumes that the tangential velocity of the fluid

* Corresponding author. Tel.: +91 712 2801407; fax: +91 712 2223230.
E-mail address: m_devakar@yahoo.co.in (M. Devakar).

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relative to the solid at a point on its surface is proportional to the tangential stress acting at that point. A review of experimental studies regarding the slip of Newtonian fluids at solid interface is given by Neto et al. [15].

The study of creeping flows of non-Newtonian fluids has gained increasing interest due to its applications in engineering and industry such as material processing in chemical engineering and hydraulic fracturing in oil recovery [16–18]. The pressure driven flow or Poiseuille flow is one of the most commonly encountered creeping flow, which has enormous applications in polymer processing such as extraction and die flow, injection molding, blow molding and asthenospheric flows [19]. Although pressure driven flows are unidirectional, and have been studied earlier for Newtonian and some non-Newtonian fluids, they still attract special attention in number of emerging problems [20–23]. In view of this, several researchers have studied the pressure driven flows for diverse non-Newtonian fluids. Ellahi [11] has examined the effect of the slip condition on flow of an Oldroyd 8-constant fluid in a channel. Yang and Zhu [24] have analyzed the squeeze flow of Bingham fluid in the small gap between parallel disks with slip boundary condition. Chen and Zhu [25] obtained the analytical solution of Couette–Poiseuille flow of Bingham fluids between two porous parallel plates with slip conditions. Hron et al. [26] established closed form analytical solution for the flows of incompressible non-Newtonian fluids with Navier's slip conditions at the boundary. Hayat et al. [27] discussed the effect of the slip condition on the flows of an Oldroyd 6-constant fluid between parallel plates. Abelman et al. [28] have analyzed the steady rotating flow of incompressible third grade fluid due to suddenly moved lower plate with partial slip of the fluid on the plate. Ellahi et al. [29] found the exact solutions for three fundamental flows namely Couette, Poiseuille and generalized Couette flows, under nonlinear slip conditions. Ferrás et al. [30] presented analytical solutions for both Newtonian and inelastic non-Newtonian fluids with slip boundary conditions in Couette and Poiseuille flows. Khaled and Vafai [31] have obtained the exact solutions of Stokes and Couette flows due to an oscillating wall with slip conditions. To the best knowledge of the authors, the Poiseuille, Couette and generalized Couette flows of couple stress fluid between parallel plates have not been solved subject to slip boundary conditions.

The aim of present communication is to establish the analytical solutions for the three classical flow problems namely *Poiseuille, Couette and generalized Couette flows* of an incompressible couple stress fluid between parallel plates under slip boundary conditions. The slip boundary conditions are applied at the boundaries of upper and lower plates. Three cases have been discussed: In the first case it is assumed that the lower and upper plates are moving with different translatory constant velocities assuming the pressure is constant (the resultant flow is known as Couette flow), in the second case both lower and upper plates are at rest and the flow is due to the constant pressure gradient which is also known as Poiseuille flow while in the last case it is assumed that the flow is generated due to the translatory motion of the upper plate when the lower plate is at rest and simultaneously a constant pressure gradient is applied. The flow in the last case known to be generalized Couette flow or Couette–Poiseuille flow. The paper is organized in terms of four sections. The next section is devoted for presenting the formulation of the basic equations governing the flow of couple stress fluid in Cartesian coordinates for the presently considered flows followed by their solutions. The results are discussed in Section 3 and the concluding remarks are presented in the last section.

2. Governing equations, problem formulation and their solutions

The equations governing the flow of a couple stress fluid are given by [2]

$$\frac{d\rho}{dt} + \rho\nabla \bullet \bar{q} = 0 \tag{2.1}$$

$$\rho\frac{d\bar{q}}{dt} = \rho\bar{f} + \frac{1}{2}\nabla \times (\rho\bar{c}) - \nabla p - \mu\nabla \times \nabla \times \bar{q} - \eta\nabla \times \nabla \times \nabla \times \nabla \times \bar{q} + (\lambda_1 + 2\mu)\nabla(\nabla \bullet \bar{q}) \tag{2.2}$$

where $\bar{q}$ and ρ are the velocity and the density of the fluid respectively, p is the fluid pressure at any point, $\bar{f}$ and $\bar{c}$ are the body force per unit mass and body couple per unit mass respectively.

The constitutive equation connecting the force stress tensor t_{ij} and rate of deformation tensor d_{ij} is given by,

$$t_{ij} = (-p + \lambda_1\nabla \bullet \bar{q})\delta_{ij} + 2\mu d_{ij} - \frac{1}{2}\varepsilon_{ijk}[m_{,k} + 4\eta\omega_{k,rr} + \rho c_k]$$

The couple stress tensor m_{ij} that arises in the theory has the linear constitutive relation

$$m_{ij} = \frac{1}{3}m\delta_{ij} + 4\eta'\omega_{j,i} + 4\eta\omega_{i,j}$$

In the above $\omega = \frac{1}{2}\nabla \times \bar{q}$ is the spin vector, $\omega_{i,j}$ is the spin tensor, d_{ij} is the rate of deformation tensor, m is the trace of couple stress tensor m_{ij} and ρc_k is the body couple vector. The quantities λ_1 and μ are the viscosity coefficients and η and η' are the couple stress viscosity coefficients. These material constants are constrained by the inequalities,

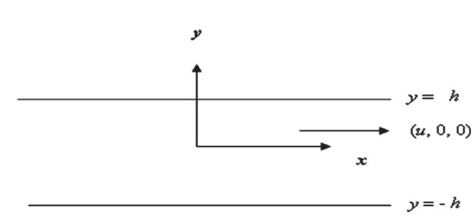
$$\mu \geqslant 0;\ 2\mu + 3\lambda_1 \geqslant 0;\ \eta \geqslant 0;\ |\eta'| \leqslant \eta$$

In the absence of body couples, the field equations governing the flow of an incompressible couple stress fluid are

$$\nabla \bullet \bar{q} = 0 \tag{2.3}$$

$$\rho\frac{d\bar{q}}{dt} = \rho\bar{f} - \nabla p - \mu\nabla \times \nabla \times \bar{q} - \eta\nabla \times \nabla \times \nabla \times \nabla \times \bar{q} \tag{2.4}$$

where $\bar{q}$ and ρ are the velocity and the density of the fluid respectively, p is the fluid pressure at any point and $\bar{f}$ is the body force per unit mass. μ and η are respectively viscosity and couple stress viscosity coefficients.



Flow Configuration

For unidirectional steady flow between parallel plates, the velocity field $\bar{q} = (u(y), 0, 0)$ and it automatically satisfies the continuity Eq. (2.3). The momentum Eq. (2.4) governing the flow, in the absence of body forces, reduces to

$$\eta \frac{d^4u}{dy^4} - \mu \frac{d^2u}{dy^2} + \frac{dp}{dx} = 0 \tag{2.5}$$

2.1. Plane Couette flow

Consider the steady laminar flow of an incompressible couple stress fluid between two infinite horizontal parallel plates distance $2h$ apart. In the absence of pressure gradient (the pressure p is assumed to be constant) no flow occurs between parallel plates and hence the velocity is zero in the flow field. The flow is due to the motion of the plates. The upper and lower plates are allowed to move with constant velocities U_1 and U_2 respectively. Here, we assume that the relative velocity between fluid and plates is proportional to the shear rate of the plates. With these, the governing Eq. (2.5) takes the form,

$$\eta \frac{d^4u}{dy^4} - \mu \frac{d^2u}{dy^2} = 0 \tag{2.6}$$

with the boundary conditions:

$$\text{(slip boundary conditions)} : u(-h) - \beta\left[\frac{du}{dy} - \frac{\eta}{\mu}\frac{d^3u}{dy^3}\right]_{y=-h} = U_2,\ u(h) + \beta\left[\frac{du}{dy} - \frac{\eta}{\mu}\frac{d^3u}{dy^3}\right]_{y=h} = U_1, \tag{2.7}$$

(vanishing of couple stresses on the boundary) :

$$\frac{d^2u}{dy^2} = 0 \text{ at } y = -h \text{ and } y = h, \tag{2.8}$$

where β is slip constant.

Introducing the following non-dimensional parameters

$$y^* = \frac{y}{h},\ u^* = \frac{u}{U_1},\ a = \sqrt{\frac{\eta}{\mu h^2}} \text{ and } \alpha = \frac{\beta}{h},$$

the boundary value problem (2.6)–(2.8), after dropping '*'s, becomes

$$a^2\frac{d^4u}{dy^4} - \frac{d^2u}{dy^2} = 0 \tag{2.9}$$

with the boundary conditions

$$\text{(slip boundary conditions)} : u(-1) - \alpha\left[\frac{du}{dy} - a^2\frac{d^3u}{dy^3}\right]_{y=-1} = d,\ u(1) + \alpha\left[\frac{du}{dy} - a^2\frac{d^3u}{dy^3}\right]_{y=1} = 1, \tag{2.10}$$

where $d = \frac{U_2}{U_1}$.

(vanishing of couple stresses on the boundary) :

$$\frac{d^2u}{dy^2} = 0 \text{ at } y = -1 \text{ and } y = 1. \tag{2.11}$$

The analytical solution of this boundary value problem (2.9)–(2.11), is given by

$$u(y) = \frac{1}{2}\left(1 + \frac{y}{1+\alpha}\right) + \frac{d}{2}\left(1 - \frac{y}{1+\alpha}\right). \tag{2.12}$$

The non-dimensional volume flow rate of the channel is given by

$$q = \int_{-1}^{1} u(y)dy = 1 + d. \tag{2.13}$$

It is noted that the solution to the steady Couette flow in the case of couple stress fluid is identical to that of viscous Newtonian fluid.

2.2. Plane Poiseuille flow

Here, the couple stress fluid is between two infinitely long horizontal parallel plates $y = -h$ and $y = h$. Both the plates are assumed to be at rest and the flow is due to the constant pressure gradient G in the positive x-direction. The equation governing this flow of incompressible couple stress fluid with slip boundary conditions is given by

$$\eta \frac{d^4u}{dy^4} - \mu \frac{d^2u}{dy^2} + \frac{dp}{dx} = 0 \tag{2.14}$$

with the conditions:

$$\text{(slip boundary conditions)} : u(-h) - \beta\left[\frac{du}{dy} - \frac{\eta}{\mu}\frac{d^3u}{dy^3}\right]_{y=-h} = 0,\ u(h) + \beta\left[\frac{du}{dy} - \frac{\eta}{\mu}\frac{d^3u}{dy^3}\right]_{y=h} = 0, \tag{2.15}$$

(vanishing of couple stresses on the boundary) :

$$\frac{d^2u}{dy^2} = 0 \ at\ y = -h \text{ and } y = h, \tag{2.16}$$

where β is slip constant.

Introducing the following non-dimensional parameters

$$y^* = \frac{y}{h},\ u^* = \frac{\rho h}{\mu}u,\ p^* = \frac{\rho h^2}{\mu^2}p,\ a = \sqrt{\frac{\eta}{\mu h^2}} \text{ and } \alpha = \frac{\beta}{h},$$

the boundary value problem (2.14)–(2.16), after dropping '*'s and using $-\frac{dp}{dx} = G$, reduce to

$$a^2\frac{d^4u}{dy^4} - \frac{d^2u}{dy^2} = G \tag{2.17}$$

subject to the boundary conditions

$$\text{(slip boundary conditions)} : u(-1) - \alpha\left[\frac{du}{dy} - a^2\frac{d^3u}{dy^3}\right]_{y=-1} = 0,\ u(1) + \alpha\left[\frac{du}{dy} - a^2\frac{d^3u}{dy^3}\right]_{y=1} = 0, \tag{2.18}$$

(vanishing of couple stresses on the boundary) :

$$\frac{d^2u}{dy^2} = 0 \text{ at} y = -1 \text{ and } y = 1. \tag{2.19}$$

Now the exact solution of this boundary value problem, after a straightforward calculation, obtained as,

$$u(y) = \frac{G}{2}[1 + 2\alpha - y^2] - \frac{G}{b^2}\left[1 - \frac{\cosh(by)}{\cosh b}\right] \tag{2.20}$$

where $b = \frac{1}{a} = \sqrt{\frac{\mu h^2}{\eta}}$.

As α approaches to zero, we obtain the solution that corresponds to the no-slip condition as

$$u(y) = \frac{G}{2}[1 - y^2] - \frac{G}{b^2}\left[1 - \frac{\cosh(by)}{\cosh b}\right] \tag{2.21}$$

which is in well agreement with the solution of the same problem with no-slip condition (page no. 44 of [2]).

The non-dimensional volume flow rate of the channel is given by

$$q = \int_{-1}^{1} u(y)dy = \frac{2G}{3} + 2G\alpha - \frac{2G}{b^2}\left(1 - \frac{\sinh b}{b \cosh b}\right). \tag{2.22}$$

2.3. Generalized Couette flow

In this case the fluid is bounded by two infinitely long horizontal parallel plates $y = -h$ and $y = h$. Here the physical model is similar to that of the Couette flow. Additionally a constant pressure gradient G is applied simultaneously in the positive x-direction. The governing equations for this flow and the boundary conditions are,

$$\eta \frac{d^4u}{dy^4} - \mu \frac{d^2u}{dy^2} - G = 0 \tag{2.23}$$

subject to the conditions:

$$\text{(slip boundary conditions)} : u(-h) - \beta\left[\frac{du}{dy} - \frac{\eta}{\mu}\frac{d^3u}{dy^3}\right]_{y=-h} = 0,\ u(h) + \beta\left[\frac{du}{dy} - \frac{\eta}{\mu}\frac{d^3u}{dy^3}\right]_{y=h} = U, \tag{2.24}$$

(vanishing of couple stresses on the boundary) :

$$\frac{d^2u}{dy^2} = 0 \text{ at } y = -h \text{ and } y = h, \tag{2.25}$$

where β is slip constant.

Making use of the following non-dimensional parameters

$$y^* = \frac{y}{h},\ u^* = \frac{u}{U}, G^* = \frac{h^2}{\mu U}G, a = \sqrt{\frac{\eta}{\mu h^2}} \text{ and } \alpha = \frac{\beta}{h},$$

Table 1 Solutions for (i) plane Couette flow, (ii) plane Poiseuille flow and (iii) generalized Couette flow.

	Couple stress fluid model	Classical Newtonian fluid model
(i)	$u(y) = \frac{1}{2}(1 + \frac{y}{1+\alpha}) + \frac{d}{2}\left(1 - \frac{y}{1+\alpha}\right)$	$u(y) = \frac{1}{2}(1 + \frac{y}{1+\alpha}) + \frac{d}{2}\left(1 - \frac{y}{1+\alpha}\right)$
(ii)	$u(y) = \frac{G}{2}[1 + 2\alpha - y^2] - \frac{G}{b^2}\left[1 - \frac{\cosh by}{\cosh b}\right]$	$u(y) = \frac{G}{2}[1 + 2\alpha - y^2]$
(iii)	$u(y) = \frac{G}{2}[1 - y^2] + \frac{1}{2}[1 + 2\alpha G + \frac{y}{(1+\alpha)}] - \frac{G}{b^2}\left[1 - \frac{\cosh by}{\cosh b}\right]$	$u(y) = \frac{G}{2}[1 - y^2] + \frac{1}{2}\left[1 + 2\alpha G + \frac{y}{(1+\alpha)}\right]$

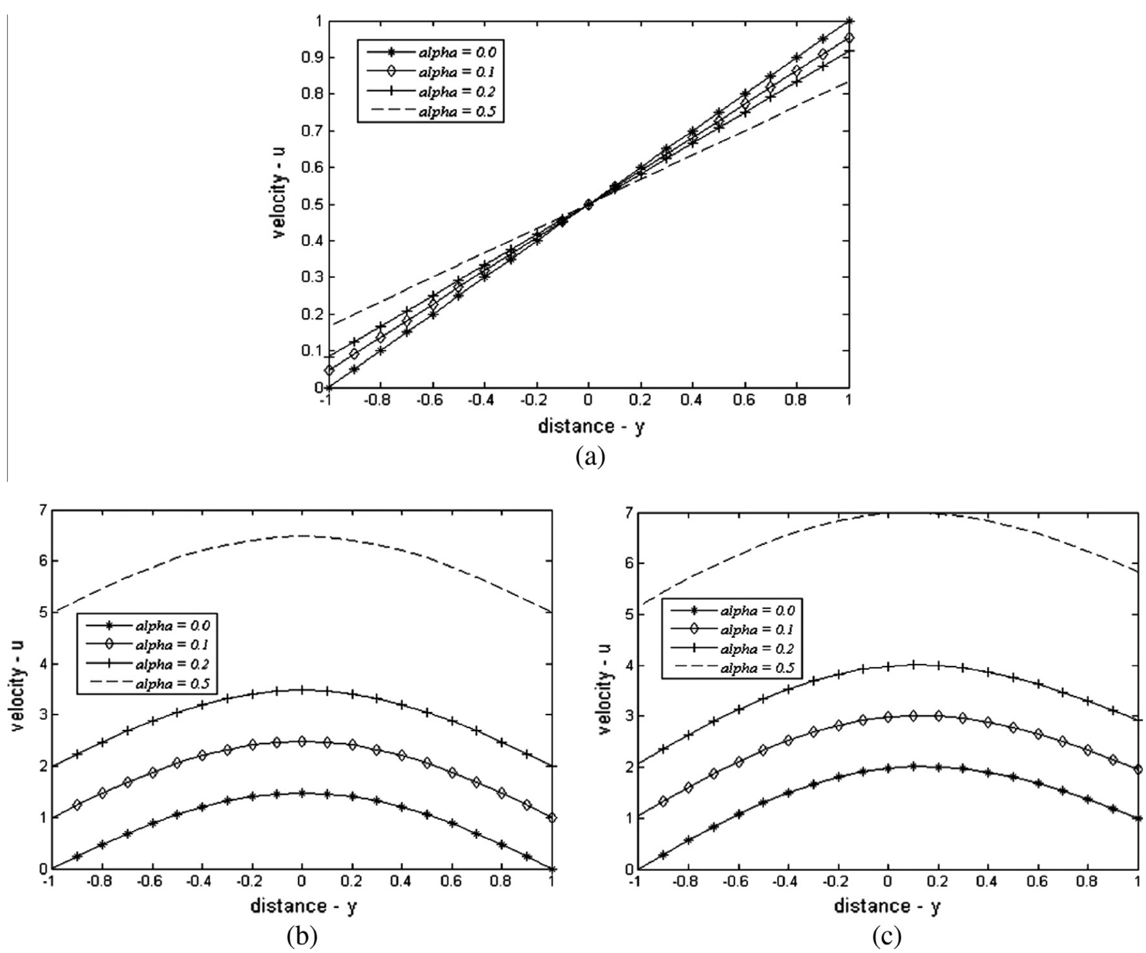


Figure 1 Velocity profiles for: (a) plane Couette flow, (b) plane Poiseuille flow and (c) generalized Couette flow with various values of non-dimensional slip parameter $\alpha(\alpha)$ for fixed values of G and a.

the boundary value problem (2.23)–(2.25), after dropping '*'s and using $-\frac{dp}{dx} = G$, becomes

$$a^2 \frac{d^4u}{dy^4} - \frac{d^2u}{dy^2} = G \tag{2.26}$$

subject to the boundary conditions

$$\text{(slip boundary conditions)} : u(-1) - \alpha\left[\frac{du}{dy} - a^2\frac{d^3u}{dy^3}\right]_{y=-1} = 0,\ \ u(1) + \alpha\left[\frac{du}{dy} - a^2\frac{d^3u}{dy^3}\right]_{y=1} = 1, \tag{2.27}$$

(vanishing of couple stresses on the boundary) :

$$\frac{d^2u}{dy^2} = 0 \text{ at} y = -1 \text{ and } y = 1. \tag{2.28}$$

Now the analytical expression of the velocity field, solving the boundary value problem (2.26)–(2.28), is given by

$$u(y) = \frac{G}{2}[1 - y^2] + \frac{1}{2}\left[1 + 2\alpha G + \frac{y}{(1+\alpha)}\right] - \frac{G}{b^2}\left[1 - \frac{\cosh by}{\cosh b}\right] \tag{2.29}$$

where $b = \frac{1}{a} = \sqrt{\frac{\mu h^2}{\eta}}$.

The non-dimensional volume flow rate of the channel, in this case, is given by

$$q = \int_{-1}^{1} u(y)dy = \frac{2G}{3} + 1 + 2G\alpha - \frac{2G}{b^2}\left(1 - \frac{\sinh b}{b\cosh b}\right). \tag{2.30}$$

3. Results and discussions

In the absence of couple stresses, that is, as $a \to 0$ or $b \to \infty$, the solutions to the problems considered in the paper are given in Table 1. It is noted that these limiting solutions are well in agreement with the solutions of respective problems obtained by using the classical viscous Newtonian model.

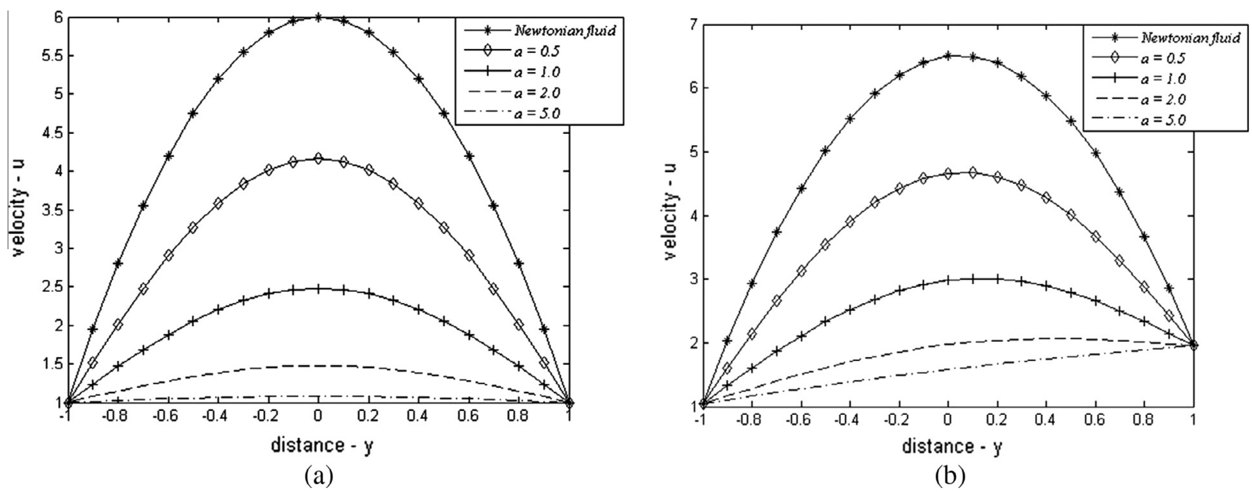


Figure 2 Velocity profiles for: (a) plane Poiseuille flow and (b) generalized Couette flow with various values of couple stress parameter a for fixed values of the slip parameter α and pressure gradient G.

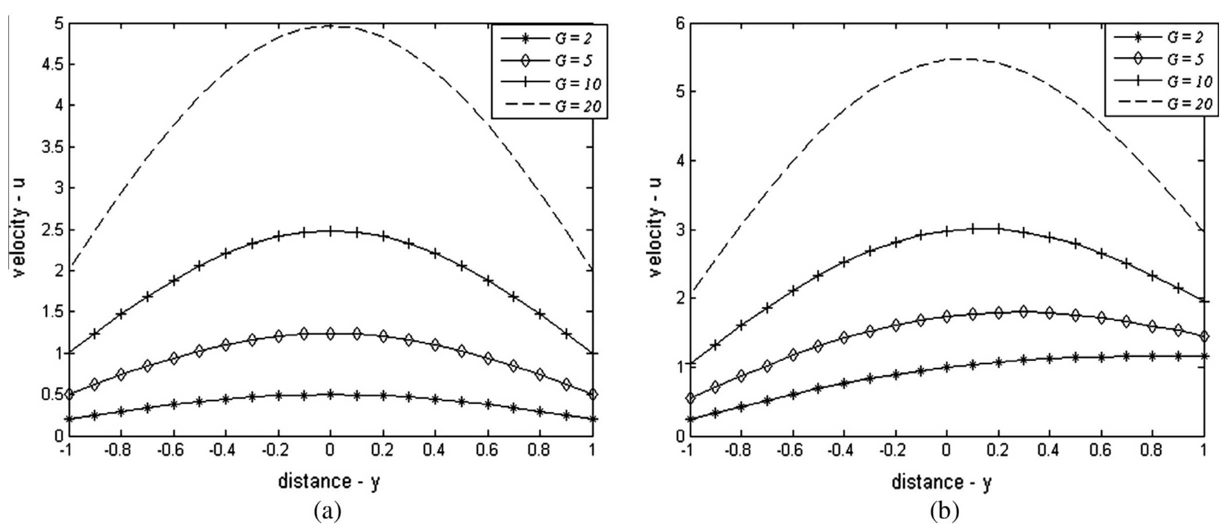


Figure 3 Velocity profiles for: (a) plane Poiseuille flow and (b) generalized Couette flow with various values of pressure gradient G when the slip parameter α and the couple stress parameter a are fixed.

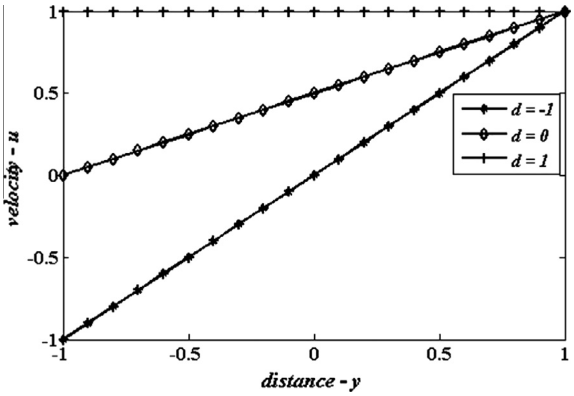


Figure 4 Velocity profiles for Couette flow with various values of velocity ratio d.

The velocity profiles for different flow parameters are plotted and the effect of slip parameter, couple stress parameter and the pressure gradient on velocity and volume flow rate for each of the problems is described.

Fig. 1 shows variation of velocity for various values of the slip parameter while the other parameters are fixed. It is observed that, for plane Couette flow, as the slip parameter increases the velocity increases near the stationary plate while the trend is reversed near the moving plate. In the other two flow situations, it is found that the increasing of slip parameter has an increasing effect on the velocity. That is, the more the fluid slips at the boundary the less its velocity affected by the motion of the boundary.

Fig. 2 depicts the variation of velocity for different values of couple stress parameter a when the other parameters held fixed for plane Poiseuille flow and for generalized Couette flow. The graphs indicate that, for both the problems, as the couple stress parameter a increases there is a decrease in the velocity

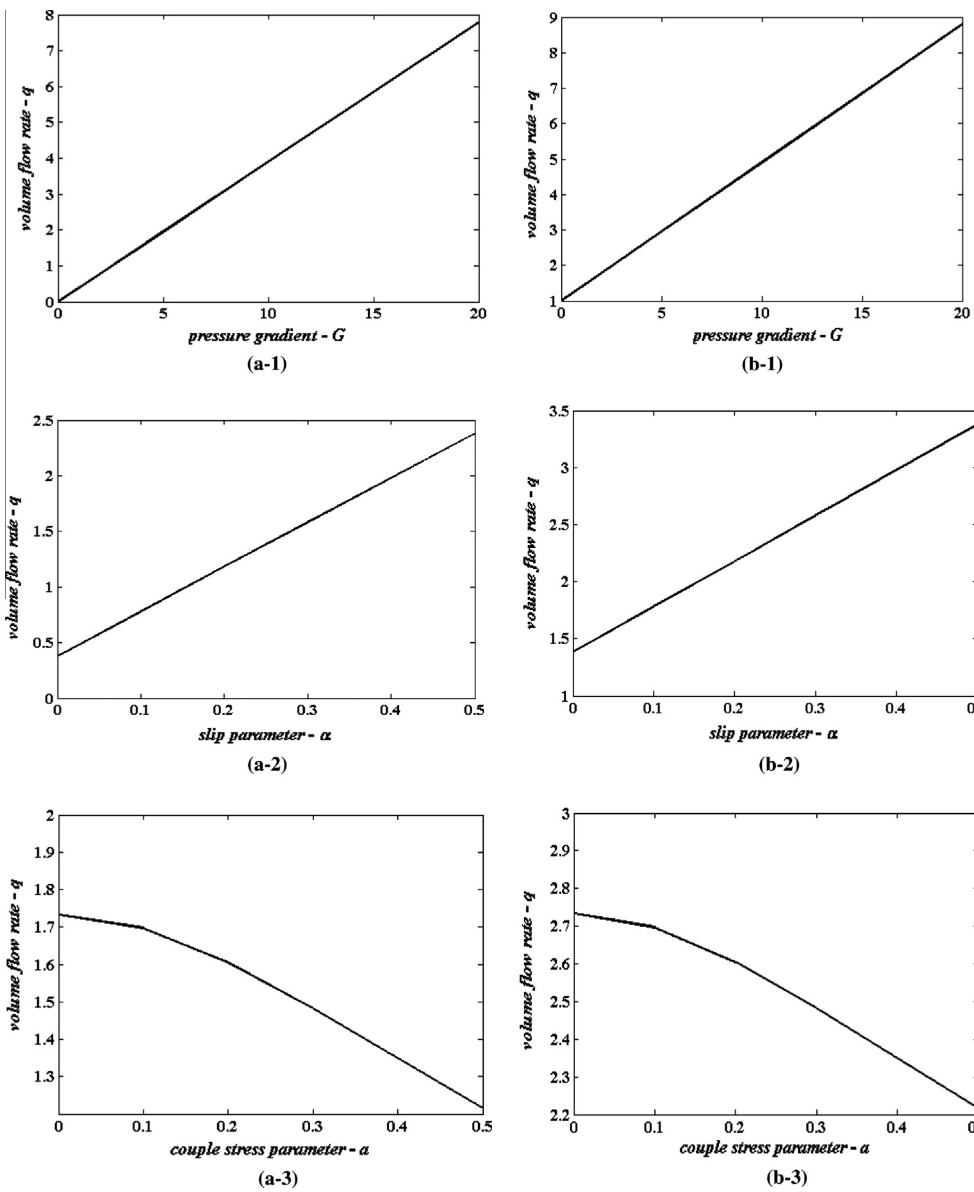


Figure 5 Volume flow rate for: (a) plane Poiseuille flow and (b) generalized Couette flow with one of the parameters when the remaining parameters held fixed.

which shows that increasing of couple stresses has decreasing effect on the velocity.

Fig. 3 shows the variation of velocity for different values of pressure gradient G for plane Poiseuille flow and generalized Couette flow. As expected, increase in the pressure gradient between parallel plates has corresponding increase in the velocity of the fluid for both the problems.

Fig. 4 displays the variation of fluid velocity for different values of d for Couette flow. $d = 0$ corresponds to the case of fixed stationary lower plate. $d = 1$ is for the case when both the plates are moving with the same constant velocity in the same direction while $d = -1$ represents the case when both the plates are moving with the same velocity in the opposite direction. From Fig. 4, it can be seen that the velocity increases when the lower plate moves in the same direction of the upper plate.

The variation of volume flow rate for various flow parameters for Poiseuille and generalized Couette flows is displayed in Fig. 5. It is noticed that, for both Poiseuille and generalized Couette flow problems, the volume flow rate of the channel increases with the increasing of pressure gradient and slip parameter while it decreases with the increasing of couple stress parameter. However, no flow parameter affects the volume flow rate in the case of plane Couette flow.

4. Concluding remarks

Three classical flow problems have been studied for an incompressible couple stress fluid between parallel plates with slip boundary conditions. The exact solution of each of the problems is obtained in an elegant way. The solutions for the limiting case as $a \to 0$ or $b \to \infty$ (as couple stresses approaches to zero) are obtained for each of the problems. It is interesting to note that these limiting solutions are well in agreement with the solutions of respective problems of Newtonian fluid. Following are the observations of the present investigation:

(i) The solution to Couette flow for couple stress fluid is same as that of Newtonian fluid.
(ii) The velocity in Couette flow increases significantly when the lower plate moves in the direction of the upper plate.
(iii) The presence of couple stresses has a decreasing effect on the fluid velocity and the volume flow rate.
(iv) More the fluid slips at the boundary (that is, as the slip parameter increases), less its velocity affected by the boundary.

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